Lecture Notes

[UG Mathematics Course under CBCS Curriculum in India]

on

Foundation Course in Algebra I

November 23, 2018

Topics Covered:

↬ Complex numbers
↬ Theory of equations
↬ Inequalities

Disclaimer: These handouts have been prepared by Dr. Subhajit Saha, Assistant Professor & Head at the Dept. of Mathematics, Panihati Mahavidyalaya. They are aimed at the Undergraduate Students in Mathematics studying under the CBCS Curriculum in Indian Universities/Colleges. The readers should, however, keep in mind that **these lecture notes are not supposed to be replacements for textbooks**. Anybody taking this course should go through the proofs of all the theorems and propositions occurring in this article and work out all the problems (examples as well as exercises) of as many textbooks as possible on every topic in order to have a firm understanding of the concepts outlined in this course. Any comments, criticisms or suggestions on the content of this course are welcome and should be directed to `subhajit1729@gmail.com`.

Contents

Course Outline:

1. **Complex numbers:** Polar representation of complex numbers, nth roots of unity, De Moivre's theorem for rational indices and its applications.
2. **Theory of equations:** Relation between roots and coefficients, transformation of equation, Descartes rule of signs, cubic and biquadratic equation.
3. **Inequalities:** The inequality involving AM $\geq$ GM $\geq$ HM, Cauchy-Schwartz inequality.

Recommended Book(s):

1. S. K. Mapa, Higher Algebra (Classical), Sarat Book Distributors (2007).
2. M. Spiegel, J. Schiller, and S. Lipschutz, Schaum's Outline of Complex Variables (2nd Ed.), The McGraw Hill Companies (2009).

References:

1. T. Andreescu and D. Andrica, Complex Numbers from A to Z, Birkhauser (2006).
2. W. J. Gilbert and S. A. Vanstone, Higher Algebra, Waterloo Mathematics Foundation (1993).
3. R. Earl, "Week 4 – Complex Numbers" (Lecture Notes), Oxford University (2003).
4. P. Neumann, "Complex Numbers" (Lecture Notes), Oxford University (2016).
5. P. Shunmugaraj, "Lecture 1 – Complex Numbers and Complex Differentiation" (MTH102N Lecture Notes), IIT Kanpur (2011).
6. B. Poonen, "7. Complex Numbers" (18.03 Lecture Notes, Spring 2014), MIT (2014).
7. J. K. Langley, "G12CAN Complex Analysis" (Lecture Notes), University of Nottingham.

1 Complex Numbers

1.1 Motivation: The Need for Complex Numbers

All of us know that the two roots of the quadratic equation $ax^2 + bx + c = 0$ with real coefficients are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

which is known as the quadratic formula or the Sridharacharya formula for solving quadratic equations. Solving quadratic equations is something that mathematicians have been able to do since the time of the Babylonians. When $b^2 - 4ac > 0$, these two roots are real and distinct; graphically they are the points where the curve $y = ax^2 + bx + c$ cuts the x-axis. When $b^2 - 4ac = 0$, we have only one real root and it is the point where the curve just touches the x-axis. But when $b^2 - 4ac < 0$, there are no real solutions to the equations . From the graphical point of view, the curve $y = ax^2 + bx + c$ lies entirely above or below the x-axis. It is only comparatively recently that mathematicians have been comfortable with these roots when $b^2 - 4ac < 0$. During the Renaissance, quadratic equations would have been considered unsolvable or its roots would have been called *imaginary.* (The term imaginary was first used by the French Mathematician René Descartes (1596 – 1650). Whilst he is known more as a philosopher, Descartes made many important contributions to mathematics and helped found co-ordinate geometry — hence the naming of Cartesian co-ordinates.) If we imagine $\sqrt{-1}$ to exist, and that it behaves (adds and multiplies) much the same as other numbers then the two roots of the quadratic can be written in the form

$$x = A + B\sqrt{-1},$$

where

$$A = -\frac{b}{2a} \quad \text{and} \quad B = \frac{\sqrt{4ac - b^2}}{2a} \text{ are real numbers.}$$

But what meaning can such roots have? It was this philosophical point which pre-occupied mathematicians until the start of the 19th century when these imaginary numbers started proving so useful (especially in the work of Cauchy and Gauss) that essentially the philosophical concerns just got forgotten about.

1.2 Cartesian Representation of Complex Numbers

This section underlines the preliminary concepts related to complex numbers in Cartesian coordinates (x, y). It is assumed that the definition and basic properties of the set of real numbers $\mathbb{R}$ are known.

1.2.1 Complex Numbers

Definition: A *complex number* is a number of the form $x + yi$, where x and y are real numbers and i is the imaginary unit equal to $\sqrt{-1}$.

If $z = x + yi$, then x is known as the real part of z and y as the imaginary part. We write $x = \text{Re } z$ and $y = \text{Im } z$. This is called the *cartesian representation* of complex numbers. Note that all real numbers are complex — a real number is simply a complex number with no imaginary part. The term 'complex number' is due to the German mathematician Carl Friedrich Gauss (1777–1855). Gauss is considered by many as the greatest mathematician ever. He made major contributions to almost every area of mathematics from number theory, to non-Euclidean geometry, to astronomy and magnetism. His name precedes a wealth of theorems and definitions throughout mathematics.

In component notation, $z = x + yi$ can be written as (x, y). The set of complex numbers is denoted by $\mathbb{C}$. In set builder form, $\mathbb{C}$ is expressed as

$$\mathbb{C} = \{z : z = x + yi, (x, y) \in \mathbb{R} \ \& \ i^2 = -1\}.$$

The set of all non-zero complex numbers $\mathbb{C} \setminus \{(0, 0)\}$ is denoted by $\mathbb{C}^*$. Note that sometimes we'll use the form $x + yi$, and sometimes $x + iy$. There is **no** mathematical difference. In some contexts, there is a mild psychological difference.

A note on i: This notation was first introduced by the Swiss mathematician Leonhard Euler (1707–1783). Much of our modern notation is due to him including e and π. Euler was a giant in 18th century mathematics and the most prolific mathematician ever. His most important contributions were in analysis (eg. on infinite series, calculus of variations). The study of topology arguably dates back to his solution of the Königsberg Bridge Problem. (Many books, particularly those written for engineers and physicists use j instead.)

A note on ordering in $\mathbb{C}$: One property of $\mathbb{R}$ that does not carry over to $\mathbb{C}$ is the order relation, which is the ability to say that one number is bigger than another. If we could compare the sizes of complex numbers, then either $i > 0$ or $i < 0$. However, the properties of any ordering would imply that $i^2 > 0$, and so $-1 > 0$. This contradicts our order relation in $\mathbb{R}$, and shows that there is no order relation in $\mathbb{C}$. It is therefore immaterial to talk about inequalities of complex numbers. However, we can talk about inequalities involving the real or imaginary parts of complex numbers and, as discussed in Sec. 2.4, the moduli (plural for 'modulus') of complex numbers. This is because, for any complex number, its real part, its imaginary part, and its modulus are all real numbers.

One of the first major results concerning complex numbers and which conclusively demonstrated their usefulness was proved by Gauss in 1799. From the quadratic formula, we know that all quadratic equations can be solved using complex numbers. What Gauss was the first to prove was the much more general result:

Theorem A.1: (Fundamental Theorem of Algebra) *The roots of any polynomial equation $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0$ with real (or complex) coefficients a_i are complex. That is there are n (not necessarily distinct) complex numbers $\gamma_1 \ldots \gamma_n$ such that*

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = a_n(x - \gamma_1)(x - \gamma_2)\cdots(x - \gamma_n).$$

In particular the theorem shows that an n degree polynomial has, counting multiplicities, n roots in $\mathbb{C}$.

The theorem only guarantees the existence of the roots of a polynomial somewhere in $\mathbb{C}$ unlike the quadratic formula which plainly gives us the roots. The theorem gives no hints as to where in $\mathbb{C}$ these roots are to be found.

1.2.2 Algebraic Operations on Complex Numbers

We add, subtract, multiply and divide complex numbers much as we would expect. We add and subtract complex numbers by adding their real and imaginary parts.

Consider two complex numbers $z_1 = a + bi$ and $z_2 = c + di$. Then

$$z_1 + z_2 = (a + bi) + (c + di) = (a + c) + (b + d)i,$$

$$z_1 - z_2 = (a + bi) - (c + di) = (a - c) + (b - d)i.$$

We can multiply complex numbers by expanding the brackets in the usual fashion and using $i^2 = -1$,

$$z_1 \cdot z_2 = (a + bi)(c + di) = (ac - bd) + (ad + bc)i,$$

and to divide complex numbers we note firstly that $(c+di)(c-di) = c^2+d^2$ is real. So

$$\frac{z_1}{z_2} = \frac{a + bi}{c + di} = \frac{a + bi}{c + di} \times \frac{c - di}{c - di} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i.$$

Proposition A.1: The addition of complex numbers satisfies the following properties:

1. **Commutative law:**

$$z_1 + z_2 = z_2 + z_1 \text{ for all } z_1, z_2 \in \mathbb{C}.$$

2. **Associative law:**

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3) \text{ for all } z_1, z_2, z_3 \in \mathbb{C}.$$

3. **Existence of additive identity:** There exists a unique complex number $0 = 0 + 0i$ such that

$$z + 0 = 0 + z = z \text{ for all } z \in \mathbb{C}.$$

4. **Existence of additive inverse:** For any complex number z, there exists a unique complex number $(-z)$ such that

$$z + (-z) = (-z) + z = 0.$$

Proposition A.2: The multiplication of complex numbers satisfies the following properties:

1. **Commutative law:**

$$z_1 \cdot z_2 = z_2 \cdot z_1 \text{ for all } z_1, z_2 \in \mathbb{C}.$$

2. **Associative law:**

$$(z_1 \cdot z_2) \cdot z_3 = z_1 \cdot (z_2 \cdot z_3) \text{ for all } z_1, z_2, z_3 \in \mathbb{C}.$$

3. **Existence of multiplicative identity:** There exists a unique complex number $1 = 1 + 0i$ such that

$$z \cdot 1 = 1 \cdot z = z \text{ for all } z \in \mathbb{C}.$$

4. **Existence of multiplicative inverse:** For any *non-zero* complex number z, there exists a unique *non-zero* complex number ϑ such that

$$z \cdot \vartheta = \vartheta \cdot z = 1.$$

   In this case, ϑ is called the inverse of z and is written $\vartheta = z^{-1} = \frac{1}{z}$. It is easy to verify that the inverse of $z = a + bi$, a and b not both zero simultaneously, is given by $\vartheta = \frac{1}{a+bi} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i$.

5. **Distributive law of multiplication over addition:**

$$z_1 \cdot (z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3 \text{ for all } z_1, z_2, z_3 \in \mathbb{C}.$$

An integer power of a complex number $z \in \mathbb{C}^*$ is defined by

$$z^0 = 1, \qquad z^1 = z, \qquad z^2 = z \cdot z;$$

$$z^n = \underbrace{z \cdot z \cdots z}_{n \text{ times}} \text{ for all integers } n > 0.$$

and $z^n = (z^{-1})^{-n}$ for all integers $n < 0$.

Proposition A.3: The following properties hold for all complex numbers $z, w, \in \mathbb{C}^*$ and for all integers m, n:

1. $z^m \cdot z^n = z^{m+n}$
2. $\frac{z^m}{z^n} = z^{m-n}$
3. $(z^m)^n = z^{mn}$
4. $(z \cdot w)^n = z^n \cdot w^n$
5. $\left(\frac{z}{w}\right)^n = \frac{z^n}{w^n}$.

When $z = 0$, we have $0^n = 0$ for all integers $n > 0$. Also, note that $i^n \in \{1, -1, i, -i\}$ for any integer n.

1.2.3 Conjugate of a Complex Number

Definition: Let $z = a + bi$. Then the number $a - bi$ is called the *complex conjugate* or simply *conjugate* of z and is denoted by $\bar{z}$ (or in some books by z^*).

It is well known from our high school textbooks that when the quadratic equation $Ax^2 + Bx + C = 0$ with real coefficients have complex roots, then these roots are conjugates of each other. Generally, if z_0 is a root of the polynomial $a_n z^n + a_{n-1} z^{n-1} + \cdots + a_o = 0$, where the a_i's are real, then so is its conjugate $\bar{z}_0$.

Proposition A.4: The conjugate operation satisfies the following properties. Let z, $w \in \mathbb{C}$. Then

1. $z = \bar{z}$ if and only if $z \in \mathbb{R}$.

2. $z = \bar{\bar{z}}$.

3. The number $z \cdot \bar{z}$ is a non-negative real number, i.e., $z \cdot \bar{z} \in \mathbb{R}^+$.

4. $\overline{z \pm w} = \bar{z} \pm \overline{w}$.

5. $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$.

6. $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$, $w \neq 0$

7. $\overline{z^{-1}} = (\overline{z})^{-1}$.

8. For any complex number z,

$$\mathrm{Re}(z) = \frac{z + \overline{z}}{2} \qquad \text{and} \qquad \mathrm{Im}(z) = \frac{z - \overline{z}}{2i}.$$

Note that Properties (4) and (5) above can be generalized as

$$4'.\ \overline{\left(\sum_{k=1}^{n} z_k\right)} = \sum_{k=1}^{n} \overline{z}_k \qquad \text{and} \qquad 5'.\ \overline{\left(\prod_{k=1}^{n} z_k\right)} = \prod_{k=1}^{n} \overline{z}_k$$

respectively, for all $z_k \in \mathbb{C}$, $k = 1, 2 \ldots, n$. Moreover, as a consequence of ($5'$.) and (6), we have

$$\overline{z^n} = (\overline{z})^n \text{ for any integer } n \text{ and for any } z \in \mathbb{C}.$$

1.2.4 Modulus of a Complex Number

Definition: Let $z = a+bi$. Then the number $\sqrt{a^2 + b^2}$ is called the *modulus* or the *absolute value* of z and is denoted by $|z|$.

Proposition A.5: The modulus operation satisfies the following properties. Let z, $w \in \mathbb{C}$. Then

1. $-|z| \leq \mathrm{Re}(z) \leq |z|$ and $-|z| \leq \mathrm{Im}(z) \leq |z|$.

2. $|z| \leq 0$ for all $z \in \mathbb{C}$ and $|z| = 0$ if and only if $z = 0$.

3. $|z| = |-z| = |\overline{z}|$.

4. $z \cdot \overline{z} = |z|^2$.

5. $|z \cdot w| = |z| \cdot |w|$.

6. $|z| - |w| \leq |z + w| \leq |z| + |w|$. The last inequality, known as the *triangle inequality*, becomes an equality if and only if $\mathrm{Re}(z\overline{w}) = |z||w|$. The latter is equivalent to $z = \lambda w$, where $\lambda \in \mathbb{R}^+$.

7. $||z| - |w|| \le |z - w|$.
8. $|z^{-1}| = |z|^{-1}$.
9. $|\frac{z}{w}| = \frac{|z|}{|w|}$.

Again, note that Property (5) and the last inequality in Property (6) above can be generalized as

$$5'. \left|\prod_{k=1}^{n} z_k\right| = \prod_{k=1}^{n} |z_k| \qquad \text{and} \qquad 6'. \left|\sum_{k=1}^{n} z_k\right| \le \sum_{k=1}^{n} |z_k|$$

respectively, for all $z_k \in \mathbb{C}$, $k = 1, 2 \ldots, n$. Moreover, as a consequence of ($5'$.), we have

$$|z^n| = |z|^n \text{ for any integer } n \text{ and for any } z \in \mathbb{C}.$$

1.3 Polar Representation of Complex Numbers

The real numbers are often represented on the *real line* which increase as we move from left to right (Fig. 1).

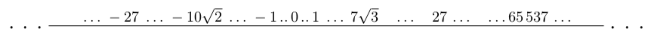


Figure 1: The real number line

The complex numbers, having two components, their real and imaginary parts, can be represented as a plane; indeed $\mathbb{C}$ is sometimes referred to as the complex plane, but more commonly when we represent $\mathbb{C}$ in this manner we call it an *Argand diagram* (Fig. 2), named after the Swiss mathematician Jean-Robert Argand (1768–1822). The point (a, b) represents the complex number $a + bi$ so that the x-axis contains all the real numbers, and so is termed the *real axis*, and the y-axis contains all those complex numbers which are purely imaginary (i.e. have no real part) and so is referred to as the *imaginary axis*. Note that the set of real numbers can also be represented as

$$\mathbb{R} = \{z \in \mathbb{C} : \text{Im } z = 0\}.$$

Note that the conjugate $\overline{z}$ of a point z is its mirror image in the real axis. So, $z \longmapsto \overline{z}$ represents reflection in the real axis.

A complex number z in the complex plane can be represented by Cartesian co-ordinates (x, y), its real and imaginary parts respectively, but it is

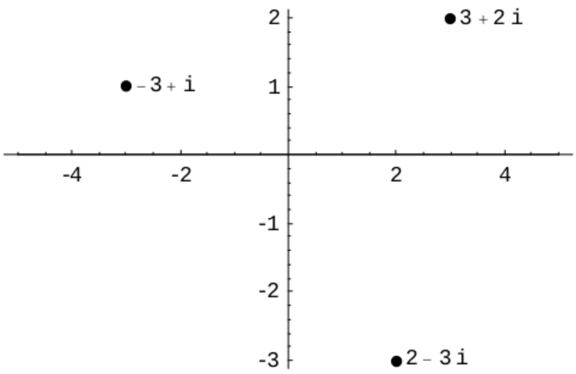


Figure 2: An Argand diagram

often useful to switch to its representation by polar co-ordinates (r, θ). If we let r be the distance of z from the origin and, if $z \neq 0$, we let θ be the angle that the line joining z to the origin makes with the positive real axis (Fig. 3), then using the *triangle law of vector addition*, we can write

$$z = x + iy = r \cos \theta + ir \sin \theta.$$

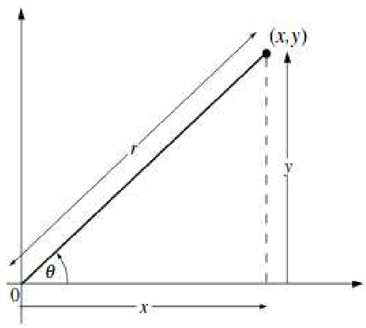


Figure 3: Polar representation of $z = x + iy$

By Euler's identity, we have $z = re^{i\theta}$. This is called the *polar representation* of complex numbers. The relations between z's Cartesian and polar co-ordinates are simple:

$$\begin{aligned} x &= r \cos \theta \ \text{ and } \ y = r \sin \theta, \\ r &= \sqrt{x^2 + y^2} \ \text{ and } \ \tan \theta = \frac{y}{x}. \end{aligned}$$

Note that r is simply the modulus of z, i.e., $|z| = r$, while the number θ is called the *argument* of z and is written arg z. It is easy to see that (1) arg 0 is undefined and (2) arg $z \pm \pi$ is an argument of $-z$.

A note on the geometric interpretation of the product of two complex numbers — zw: As a transformation of the Argand diagram, the operation 'multiply by w' leaves the point 0 unmoved, scales distances from 0 by a factor $|w|$, and rotates the plane anticlockwise about 0 through an angle arg w. To clarify, let $z = re^{i\theta}$ and $w = Re^{i\phi}$, then

$$zw = rRe^{i(\theta+\phi)} = |z||w|\ [\cos(\theta + \phi) + i\ \sin(\theta + \phi)].$$

It is worth remembering that this anticlockwise convention for rotations is standard not just in the context of the complex plane, but also in geometry, in mechanics — in most of mathematics. There is more to the convention, though: if $\alpha > 0$ then an anticlockwise rotation through angle $-\alpha$ is a clockwise rotation through angle α.

The cautious reader may have already noticed that the notation θ for arg z is ambiguous since there are many values of θ for the same z which means that arg z is not a function as such. Let us demonstrate this by an example. Suppose that $z = 3i$. So z corresponds to the point $(0, -3)$. Then $r = |z| = 3$, but there are infinitely many possibilities for the angle θ. One possibility is $-\frac{\pi}{2}$; all the others are obtained by adding integer multiples of 2π:

$$\arg z = \ldots, -\frac{5\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \ldots.$$

So z has many polar forms:

$$z = \cdots = 3e^{i(-5\pi/2)} = 3e^{i(-\pi/2)} = 3e^{i(3\pi/2)} = 3e^{i(7\pi/2)} = \cdots.$$

Thus, two polar forms $r_1e^{i\theta_1}$ and $r_2e^{i\theta_2}$ are equal if and only if $r_1 = r_2$ and $\theta_2 = \theta_1 + 2k\pi$ for some $k \in \mathbb{Z}$. Now, in order to specify a unique polar form, we have to restrict the range for θ to some interval of width 2π. The most common choice is to require $-\pi < \theta \leq \pi$ because it makes 'arg' single-valued. This special θ is called the *principal value of the argument*, and is denoted as $\theta = \mathrm{Arg}\ z$.

Thus, depending on the position of z on the Argand diagram, the value of Arg z can be evaluated on the basis of the following rule:

$$\text{Arg } z = \begin{cases} 0 & \text{for } z \text{ lying on the positive real axis} \\ \arg z & \text{for } z \text{ lying in the first quadrant} \\ \frac{\pi}{2} & \text{for } z \text{ lying on the positive imaginary axis} \\ \pi - \arg z & \text{for } z \text{ lying in the second quadrant} \\ \pi & \text{for } z \text{ lying on the negative real axis} \\ \arg z - \pi & \text{for } z \text{ lying in the third quadrant} \\ -\frac{\pi}{2} & \text{for } z \text{ lying on the negative imaginary axis} \\ -\arg z & \text{for } z \text{ lying in the fourth quadrant.} \end{cases}$$

Proposition A.6: The argument operation satisfies the following properties. Let z, $w \in \mathbb{C}^*$. Then

1. $\arg (zw) = \arg z + \arg w + 2k\pi$ for all $k \in \mathbb{Z}$.
2. $\arg(\frac{z}{w}) = \arg z - \arg w + 2k\pi$ for all $k \in \mathbb{Z}$.
3. $\arg \overline{z} = - \arg z + 2k\pi$ for all $k \in \mathbb{Z}$.

Warning: It is not always true that Arg z+Arg w = Arg (zw). For instance, if $z = w = -1+i$, then Arg z = Arg $w = \frac{\pi}{4}$ which implies Arg z+Arg $w = \frac{\pi}{2}$ but Arg $(zw) = -\frac{\pi}{2}$.

Note that Property (1) above can be generalized as

$$\arg \left(\prod_{k=1}^{n} z_k \right) = \sum_{k=1}^{n} \arg z_k$$

for all $z_k \in \mathbb{C}^*$, $k = 1, 2 \ldots, n$. This leads us to

$$\arg (z^n) = n \arg z$$

for any non-negative integer n. This relation forms the basis of a famous theorem due to de Moivre:

Theorem A.2: (De Moivre's Theorem) *For a real number θ and integer n, we have*

$$(\cos \theta \pm i \sin \theta)^n = \cos n\theta \pm i \sin n\theta.$$

When n is a fraction, positive or negative, and θ is a real number,

$$(\cos \theta \pm i \sin \theta)^n = \cos n\theta \pm i \sin n\theta.$$

is one of the values of $(\cos\theta \pm i\sin\theta)^n$.

Abraham de Moivre (1667–1754), a French protestant who moved to England, is best remembered for this formula but his major contributions were in probability and appeared in his *The Doctrine of Chances* (1718). de Moivre's Theorem has many uses. One of them is to express the cosines and sines of multiples of an angle θ as polynomials in $\cos\theta$ and $\sin\theta$.

1.4 Roots of Unity

If $z \in \mathbb{C}$, $n \in \mathbb{N}$ and $z^n = 1$, then z is known as a *root of unity* — an n^{th} *root of unity*, to be precise. For such numbers, since $|z|^n = |z^n| = 1$ and $|z| > 0$, we have $|z| = 1$.

We now define a special set:

$$S^1 = \{z \in \mathbb{C} : |z| = 1\} \qquad \text{(standard notation)}$$

which can be identified as the unit circle in the complex plane (Argand diagram). The reader will learn in a future course that this set is an example of a *group* and is known as the *circle group*.

We now apply these ideas to determine the n^{th} roots of unity. In other words, we determine all those complex numbers z which satisfy $z^n = 1$, $n \in \mathbb{N}$. Well, it is already known from the Fundamental Theorem of Algebra that there are (counting multiplicities) n solutions to the equation $z^n = 1$. Before going into the general case, let us first solve the equation for $n = 2, 3, 4$.

- When $n = 2$, we have

$$0 = z^2 - 1 = (z-1)(z+1)$$

and so $z = \pm 1$.

- When $n = 3$, we have

$$0 = z^3 - 1 = (z-1)(z^2+z+1).$$

So 1 is a root and completing the square, we see

$$0 = z^2 + z + 1 = \left(z + \frac{1}{2}\right)^2 + \frac{3}{4}$$

which has roots $-\frac{1}{2} \pm \frac{\sqrt{3}i}{2}$. So the cube roots of 1 are

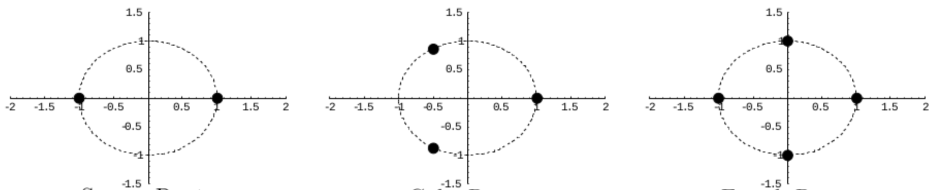
$$1, -\frac{1}{2} + \frac{\sqrt{3}i}{2}, -\frac{1}{2} - \frac{\sqrt{3}i}{2}.$$

- When $n = 4$, we can factorise as

$$0 = z^4 - 1 = (z^2 - 1)(z^2 + 1) = (z - 1)(z + 1)(z - i)(z + i),$$

so the fourth roots of 1 are $1, -1, i, -i$.

Plotting these roots on Argand diagrams, we see a pattern developing:



Square Roots Cube Roots Fourth Roots

Figure 4: Representation of square, cube, and fourth roots of unity on Argand diagrams

Returning to the general case, suppose that

$$z = r(\cos\,\theta + i\,\sin\,\theta) \;\text{ and satisfies } z^n = 1.$$

Then, z^n has modulus r^n and argument $n\theta$ whilst 1 has modulus 1 and argument 0. Comparing their moduli, we have

$$r^n = 1 \Longrightarrow r = 1.$$

Comparing arguments, we see $n\theta = 0 + 2k\pi$, $k \in \mathbb{Z}$ giving $\theta = \frac{2k\pi}{n}$. Therefore

$$z = \cos\left(\frac{2k\pi}{n}\right) + i\,\sin\left(\frac{2k\pi}{n}\right),\; k \in \mathbb{Z}.$$

At first glance, there seem to be an infinite number of roots but, as cos and sin have period 2π, these z repeat with period n. Hence we have shown that

Theorem A.3: *The n^{th} roots of unity, that is the solutions of the equation $z^n = 1$, are*

$$z = \cos\left(\frac{2k\pi}{n}\right) + i\,\sin\left(\frac{2k\pi}{n}\right),\; k = 0, 1, 2, \ldots, n-1.$$

Plotted on an Argand diagram, these n^{th} roots of unity form a regular polygon of n sides inscribed within the unit circle with a vertex at 1.

We can now easily determine all those complex numbers z such that $z^n = w$ for any $w \in \mathbb{C}$. Following the above procedure, we have that

$$z = |w|^{\frac{1}{n}} \cos\left(\frac{\arg\, z + 2k\pi}{n}\right) + i \sin\left(\arg\, z + \frac{2k\pi}{n}\right),\ k = 0, 1, 2, \ldots, n-1.$$

For instance, all the solutions to the cubic equation $z^3 = -2 + 2i$ is given by

$$z = \sqrt{2} \cos\left(\frac{\frac{3\pi}{4} + 2k\pi}{3}\right) + i \sin\left(\frac{\frac{3\pi}{4} + 2k\pi}{3}\right),\ k = 0, 1, 2.$$

So, the three roots of the given equation, corresponding to $k = 0, 1, 2$ repectively, are

$$1 + i, \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\right) + i\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right), \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\right) + i\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}\right).$$

1.5 Complex Exponential, Logarithmic, and Trigonometric Functions

1.5.1 Exponential Function

The *complex exponential function* is defined via its power series:

$$\exp\, z = e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!},$$

where z is any complex number.

Proposition A.7: The complex exponential function satisfies the following properties. Let z, z_1, z_2 be complex numbers. Then

1. $e^{z_1} \cdot e^{z_2} = e^{z_1 + z_2}$.
2. $\frac{e^{z_1}}{e^{z_2}} = e^{z_1 - z_2}$. Hence, $\frac{1}{e^{z_1}} = e^{-z_1}$.
3. $(e^z)^n = e^{nz}$, where n is an integer.
4. If n be a fraction, say $\frac{p}{q}$, $(e^z)^n$ has q distinct values but e^{nz} is unique. In this case, e^{nz} is one of the values of $(e^z)^n$.
5. $e^{2n\pi i} = 1$, where n is any integer.

6. $e^{z+2n\pi i} = e^z$. This states that the complex exponential function is periodic with period π.

If $z = x + iy$ be a complex number, then using Euler's identity, the exponential function of z is written as

$$e^z = e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos\, y + i\, \sin\, y).$$

Since $e^x > 0$ for all real x, $e^x(\cos\, y + i\, \sin\, y)$ represents a complex number in polar form, e^x being the modulus and y being an amplitude of $\exp\, z$.

Since $e^x \neq 0$ for any real number x, e^z is a non-zero complex number for any complex number z.

Let $u + iv$ be a non-zero complex number and let its polar representation be $r(\cos\, \theta + i\, \sin\, \theta)$. Since r is positive, $\log\, r$ is real and r can be expressed as $r = e^{\log\, r}$. Therefore,

$$\begin{aligned} u + iv &= e^{\log\, r}(\cos\, \theta + i\, \sin\, \theta) \\ &= e^{\log\, r} \cdot e^{i\theta} \\ &= e^{\log\, r + i\theta} \\ &= \exp(\log\, r + i\theta). \end{aligned}$$

Thus, when $u+iv$ is a given non-zero complex number, there exists a complex number $z = \log\, r + i\theta$ such that $\exp\, z = u + iv$. This means that the range of the exponential function of z is the entire complex plane excluding the origin.

1.5.2 Logarithmic Function

Let z be a non-zero complex number. Then there always exists a complex number w such that $e^w = z$. This w is said to be *a complex logarithm* of z.

The real logarithm function $\ln\, x$ is defined as the inverse of the exponential function, i.e, $y = \ln\, x$ is the unique solution of the equation $x = e^y$. This works because e^x is a one-to-one function: if $x_1 \neq x_2$, then $e^{x_1} \neq e^{x_2}$. Well, this is not the case for e^z. By Property 6 of Proposition A.7, we have $e^w = e^{w+2n\pi i}$, where n is an integer. This shows that if w is a complex logarithm of z, then $w + 2n\pi i$ is also a complex logarithm of z which means that "logarithm of z" is a many-valued function of z. This is denoted by $\log\, z = w + 2n\pi i$.

Now, to find w, we let $w = u + iv$ be the Cartesian form of w and $z = re^{i\theta}$

be the polar form of z. When we substitute these values into $z = e^w$, we have

$$re^{i\theta} = e^{u+iv} = e^u \cdot e^{iv}.$$

Equality of these complex numbers gives $e^u = r$ and $v = \theta = \arg z$. The first condition further gives $u = \ln r = \ln |z|$. Thus, $w = \ln |z| + i \arg z$ and a logarithm of the complex number z is

$$\log z = \ln |z| + i \arg z.$$

Note that we use ln only for logarithms of real numbers, while log denotes logarithms of complex numbers with base e (and no other base is used).

1.5.3 Trigonometric Function

2 Theory of Equations

Keep in mind that in this section, an equation always means a polynomial equation, unless stated otherwise.

2.1 Descarte's Rule of Signs

In a sequence of real numbers $a_0, a_1, \ldots, a_n$, none of which is zero, the signs of two consecutive terms may be same or different. When same sign occurs, we say that the elements show a *continuation* of signs; when the signs are different we say that the elements show a *variation* of signs. However, if some of the elements of a sequence be zero, we ignore their presence in the sequence and count the number of continuations and variations of signs. For example, in the sequence $1, 3, -2, 0, -3, 0, 4, 0, 0, 7,$ there are 3 continuations and 2 variations of signs.

Theorem B.1: (Descarte's Rule of Signs) *The number of positive roots of a polynomial equation $f(x) = 0$ with real coefficients does not exceed the number of variations of signs in the sequence of the coefficients of $f(x)$ and if less, it is less by an even number.*

The above theorem asserts that if v be the number of variations of signs and r be the number of positive roots, then $v = r + 2h$ where h is a non-negative integer. This remarkable rule of determining the nature of roots of a polynomial equation first appeared in Descarte's revolutionary work *La Géometrie* in 1637.

Corollary B.1.1: *The number of negative roots of a polynomial equation $f(x) = 0$ with real coefficients does not exceed the number of variations of signs in the sequence of coefficients of $f(-x)$ and if less, it is less by an even number.*

Corollary B.1.2: *If $f(x) = 0$ be a polynomial equation of degree n with real coefficients having no zero root and v, v' are respectively the number of variations of signs in the sequence of coefficients of $f(x)$ and $f(-x)$ such that $v + v' < n$, then the equation $f(x) = 0$ has at least $n - (v + v')$ complex roots.*

Corollary B.1.3: *If all the roots of the polynomial equation $f(x) = 0$ be non-zero real and v, v' are respectively the number of variations of signs in the sequence of coefficients of $f(x)$ and $f(-x)$, then the equation $f(x) = 0$ has v positive roots and v' negative roots.*

2.2 Relation between Roots and Coefficients

Let $f(x) = a_o x^n + a_1 x^{n-1} + \cdots + a_n$ be a polynomial of degree n with coefficients real or complex. Then $a_0 \neq 0$. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of the equation $f(x) = 0$. Then

$$\begin{aligned} a_o x^n + a_1 x^{n-1} + \cdots + a_n &= a_0(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n) \\ &= a_0[x^n - \Sigma\alpha x^{n-1} + \Sigma\alpha_1\alpha_2 x^{n-2} - \cdots + (-1)^n(\alpha_1\alpha_2\cdots\alpha_n)], \end{aligned}$$

where

$$\begin{aligned} \Sigma\alpha_1 &= \text{sum of the roots} \\ \Sigma\alpha_1\alpha_2 &= \text{sum of the product of the roots taken two at a time} \\ \cdots & \quad \cdots \\ \Sigma\alpha_1\alpha_2\cdots\alpha_r &= \text{sum of the products of the roots taken r at a time.} \end{aligned}$$

From the equality of polynomials, it follows that

$$\begin{aligned} a_1 &= a_0(-\Sigma\alpha_1) \\ a_2 &= a_0(\Sigma\alpha_1\alpha_2) \\ \cdots & \quad \cdots \\ a_n &= a_0(-1)^n(\alpha_1\alpha_2\cdots\alpha_n). \end{aligned}$$

Therefore,

$$\begin{aligned} \Sigma\alpha_1 &= -\frac{a_1}{a_0} \\ \Sigma\alpha_1\alpha_2 &= \frac{a_2}{a_0} \\ \cdots & \quad \cdots \\ \Sigma\alpha_1\alpha_2\cdots\alpha_r &= (-1)^r\frac{a_r}{a_0}. \end{aligned}$$

For example, if $\alpha, \beta, \gamma, \delta$ are the roots of the equation $a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0$, then

$$\begin{aligned} \alpha+\beta+\gamma+\delta &= -\frac{a_1}{a_0} \\ \alpha\beta+\beta\gamma+\gamma\delta+\delta\alpha &= \frac{a_2}{a_0} \\ \alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta &= -\frac{a_3}{a_0} \\ \alpha\beta\gamma\delta &= \frac{a_4}{a_0}. \end{aligned}$$

2.3 Symmetric Functions of Roots

A function f of two or more variables is said to be a *symmetric function* if f remains unaltered by an interchange of any two of its variables. For example, $f(x, y, z) = x^2y^2 + y^2z^2 + z^2x^2$ is a symmetric function of x, y, z. $f(x, y, z) = xy + yz$ is not symmetric in x, y, z, because f does not remain unaltered if x and y are interchanged.

A symmetric function of the roots of an equation is an expression that involves all the roots alike and the expression remains unaltered if any two of the roots be interchanged. For example, if α, β, γ be the roots of the equation, then $\alpha^2 + \beta^2 + \gamma^2$ is a symmetric function of the roots, while $\alpha^2\beta + \beta^2\gamma + \gamma^2\alpha$ is not a symmetric function.

A symmetric function which is the sum of a number of terms of the same type is represented by any one of its terms with a Σ before it. For example, the symmetric function $\alpha^2 + \beta^2 + \gamma^2$ is represented by $\Sigma\alpha^2$, while $\alpha^2\beta\gamma + \beta^2\gamma\alpha + \gamma^2\alpha\beta$ is represented by $\alpha^2\beta\gamma$.

Theorem B.2: (Newton's Theorem) *Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of the equation*

$$f(x) = x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_n = 0$$

and let $s_r = \alpha_1^r + \alpha_2^r + \cdots + \alpha_n^r$, where r is an non-negative integer. Then
1. $s_r + p_1s_{r-1} + p_2s_{r-2} + \cdots + p_{r-1}s_1 + rp_r = 0.$ *if* $1 \le r < n$
2. $s_r + p_1s_{r-1} + p_2s_{r-2} + \cdots + p_ns_{r-n} = 0,$ *if* $r \ge n$.

Note 1: $s_1, s_2, s_3, \ldots$ can be successively calculated in terms of the coefficients of the equation.

We have

$$\begin{aligned} s_1 + p_1 &= 0 \text{ and therefore } s_1 = p_1 \\ s_2 + p_1s_1 + 2p_2 &= 0 \text{ and therefore } s_2 = p_1^2 - 2p_2 \end{aligned}$$

and so on.

Note 2: If none of $\alpha_1, \alpha_2, \ldots, \alpha_n$ be zero, then $s_{-1}, s_{-2}, s_{-3}, \ldots$ can be calculated successively.

Let us consider the equation $x^{-1}f(x) = 0$, i.e.,

$$x^{n-1} + p_1x^{n-2} + \cdots + p_{n-1} + p_nx^{-1} = 0.$$

Putting $x = \alpha_1, \alpha_2, \ldots, \alpha_n$ in succession and adding, we have

$$s_{n-1} + p_1 s_{n-2} + \cdots + p_{n-1} n + p_n s{-1} = 0.$$

But $s_{n-1} + p_1 s_{n-2} + \cdots + (n-1)p_{n-1} = 0$.
Therefore, $p_n s_{n-1} + p_{n-1} = 0$.
This gives s_{-1}.

Again, let us consider the equation $x^{-2} f(x) = 0$, i.e.,

$$x^{n-2} + p_1 x^{n-3} + \cdots + p_{n-2} + p_{n-1} x^{-1} + p_n x^{-2} = 0.$$

Putting $x = \alpha_1, \alpha_2, \ldots, \alpha_n$ in succession and adding, we have

$$s_{n-2} + p_1 s_{n-3} + \cdots + p_{n-2} n + p_{n-1} s{-1} + p_n s_{-2} = 0.$$

But $s_{n-2} + p_1 s_{n-3} + \cdots + (n-2)p_{n-2} = 0$.
Therefore, $p_n s_{n-2} + p_{n-1} s_{-1} + 2p_{n-2} = 0$.
This gives s_{-2}.

Note 3: If $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of the equation

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \cdots + p_n = 0,$$

then $\Sigma \alpha_1^m \alpha_2^q$ can be calculated when m and q are positive integers.

When $m \neq q$,

$$\begin{aligned} \Sigma \alpha_1^m \Sigma \alpha_2^q &= \Sigma \alpha_1^{m+q} + \Sigma \alpha_1^m \alpha_2^q \\ \Longrightarrow \quad \Sigma \alpha_1^m \alpha_2^q &= s_m s_q - s_{m+q}. \end{aligned}$$

When $m = q$,

$$\begin{aligned} (\Sigma \alpha_1^m)^2 &= \Sigma \alpha_1^{2m} + 2\Sigma \alpha_1^m \alpha_2^m \\ \Longrightarrow \quad \Sigma \alpha_1^m \alpha_2^m &= \frac{1}{2}(s_m^2 - s_{2m}). \end{aligned}$$

2.4 Transformation of Equations

When a polynomial equation is given whose roots are not known, it is possible to obtain a new polynomial equation whose roots are connected with those of the given equation by some assigned relation. The method of finding the new equation is said to be a *transformation.* Such a transformation sometimes helps us to study the nature of the roots of the given equation which would have been otherrwise a difficult job.

Before taking up the general procedure, we discuss some typical transformations of polynomial equations.

2.4.1 To transform a polynomial equation whose roots are $\alpha_1, \alpha_2, \dots, \alpha_n$ into another polynomial equation whose roots are $m\alpha_1, m\alpha_2, \dots, m\alpha_n$; $m \in \mathbb{Z}$

Let the polynomial equation whose roots are $\alpha_1, \alpha_2, \dots, \alpha_n$ be

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0.$$

Let $y = m\alpha_1$. Then $\alpha_1 = \frac{y}{m}$. Since α_1 is a root of the given equation, so

$$a_0\alpha_1^n + a_1\alpha_1^{n-1} + \cdots + a_{n-1}\alpha_1 + a_n = 0.$$

Therefore,

$$a_0\left(\frac{y}{m}\right)^n + a_1\left(\frac{y}{m}\right)^{n-1} + \cdots + a_{n-1}\left(\frac{y}{m}\right) + a_n = 0.$$

or, $a_0y^n + a_1my^{n-1} + \cdots + a_{n-1}m^{n-1}y + a_nm^n = 0$. This is the transformed equation.

Note 1: The successive coefficients of the transformed equation are obtained by multiplying the successive coefficients of the given equation, beginning from the first, by $1, m, \dots, m^{n-1}, m^n$ respectively.

Note 2: This transformation is useful for the purpose of removing fractional coefficients of an equation or reducing the leading coefficient of an equation to unity.

2.4.2 To transform an a polynomial equation into one whose roots are reciprocal of the roots of the given equation

Let the polynomial equation whose roots are $\alpha_1, \alpha_2, \dots, \alpha_n$ be

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0.$$

It is assumed that none of the roots is zero. Let $y = \frac{1}{\alpha_1}$. Then $\alpha_1 = \frac{1}{y}$. Since α_1 is a root of the given equation, so

$$a_0\alpha_1^n + a_1\alpha_1^{n-1} + \cdots + a_{n-1}\alpha_1 + a_n = 0.$$

Therefore,

$$a_0\left(\frac{1}{y}\right)^n + a_1\left(\frac{1}{y}\right)^{n-1} + \cdots + a_{n-1}\left(\frac{1}{y}\right) + a_n = 0.$$

or, $a_ny^n + a_{n-1}y^{n-1} + \cdots + a_1y + a_0 = 0$. This is the transformed equation.

Note 1: The coefficients of the given equation appear in the reverse order in the transformed equation.

2.4.3 To transform a polynomial equation whose roots are $\alpha_1, \alpha_2, \ldots, \alpha_n$ into another polynomial equation whose roots are $\alpha_1 - h, \alpha_2 - h, \ldots, \alpha_n - h$; h constant

Let the polynomial equation whose roots are $\alpha_1, \alpha_2, \ldots, \alpha_n$ be

$$a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n = 0.$$

Let $y = \alpha_1 - h$. Then $\alpha_1 = y + h$. Since α_1 is a root of the given equation, so

$$a_0 (y+h)^n + a_1 (y+h)^{n-1} + \cdots + a_{n-1}(y+h) + a_n = 0.$$

This can be expressed as $A_0 x^n + A_1 x^{n-1} + \cdots + A_{n-1} x + A_n = 0$, where A_n is the remainder when $f(x)$ is divided by $x - h$ and let $q_1(x)$ be the quotient. $A_{n-1} i$ is the remainder when the quotient $q_1(x)$ is divided by $x - h$ and let $q_2(x)$ be the quotient. Repeating the same process, $A_{n-2}, A_{n-3}, \ldots, A_1$ are obtained as the successive remainders and finally $A_0 = a_0$.

Note 1: Here the transformation that is applied to the equation is given by $x = y + h$.

Note 2: The transformation can be utilized to remove a specified term from an equation.

2.4.4 Transformation in general

Given an equation $f(x) = 0$, we are to obtain an equation $\phi(y) = 0$ whose roots are connected with the roots of the given equation by a relation $\psi(x, y) = 0$. $\phi(y)$ is obtained by eliminating x between $f(x) = 0$ and $\psi(x, y) = 0$.

2.5 Reciprocal Equations

A polynomial equation is said to be a *reciprocal equation* if the reciprocal of each of its roots is also a root of it.

Therefore, a necessary condition for the polynomial equation $f(x) = 0$ to be a reciprocal equation is that 0 is not a root of it, i.e., $f(0) \neq 0$.

If $f(x) = 0$ be a reciprocal equation of degree n having roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $\phi(x) = 0$ be the polynomial equation whose roots are $\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \ldots, \frac{1}{\alpha_n}$, then the equations $f(x) = 0$ and $\phi(x) = 0$ are identical.

Let $a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$ be a reciprocal equation. Then it is identical with the equation $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$. Therefore,

$$(a_0, a_1, \ldots, a_n) = k(a_n, a_{n-1}, \ldots, a_0) \text{ for some } k \neq 0.$$

That is, $a_0 = ka_n, a_1 = ka_{n-1}, \ldots, a_n = ka_0$ which gives $k = \pm 1$. Thus, two cases arise:

Case 1 − $k = 1$: In this case $a_0 = a_n, a_1 = a_{n-1}, \ldots, a_n = a_0$. The coefficients of equidistant terms from the beginning and the end are equal in magnitude and have the same sign. The equation is said to be a reciprocal equation of the *first type* or *first class.*

Case 1 − $k = -1$: In this case $a_0 = -a_n, a_1 = -a_{n-1}, \ldots, a_n = -a_0$. The coefficients of equidistant terms from the beginning and the end are equal in magnitude but of opposite signs. The equation is said to be a reciprocal equation of the *second type* or *second class.*

Examples: 1. The equation $x^2 + 1$ is a reciprocal equation of degree 2 and of the first type.

2. The equation $3x^3 - 13x^2 + 13x - 3 = 0$ is a reciprocal equation of degree 3 and of the second type.

Theorem B.3: *If $f(x) = 0$ be a reciprocal equation of degree n and of the first type, then $f(x) = x^n f\left(\frac{1}{x}\right)$. Conversely, if $f(x)$ be a polynomial of degree n and $f(x) = x^n f\left(\frac{1}{x}\right)$, then $f(x) = 0$ is a reciprocal equation of the first type.*

Theorem B.4: *If $f(x) = 0$ be a reciprocal equation of degree n and of the second type, then $f(x) = -x^n f\left(\frac{1}{x}\right)$. Conversely, if $f(x)$ be a polynomial of degree n and $f(x) = -x^n f\left(\frac{1}{x}\right)$, then $f(x) = 0$ is a reciprocal equation of the second type.*

Theorem B.5: The solution of any reciprocal equation depends on that of a reciprocal equation of the first type and of even degree.

Definition: A reciprocal equation is said to be of the *standard form* if it is of the first type and of even degree.

Theorem B.6: The solution of a reciprocal equation of the first type and of degree $2m$ depends on that of an equation of degree m.

2.6 The Cubic Equation

2.7 The Biquadratic Equation

3 Inequalities

3.1 Basic Concepts

When two real numbers are not equal, a relation of *inequality* is said to exist between them. The *law of trichotomy* in $\mathbb{R}$ states that any two real numbers a, b must satisfy one and only one of the following relation:

1. a is greater than b ($a > b$),
2. a is equal to b ($a = b$),
3. a is less than b ($a < b$).

The first and third in the above list are inequality relations.

Therefore, if a be a real number different from 0, then one of the following inequalities must hold:

1. $a > 0$,
2. $a < 0$.

When $a > 0$, a is said to be *positive*; when $a < 0$, a is said to be *negative*.

We define $a > b$ if $a - b > 0$ and $a < b$ if $a - b < 0$. Note that the relations $a > b$ and $b < a$ state the same inequality relation, since $a > 0 \Leftrightarrow -a < 0$.

The symbol $a \geq b$ means a is greater than or equal to b or, more precisely, a is *at least* b. On the other hand, the symbol $a \leq b$ means a is less than or equal to b or, more precisely, a is *at most* b.

3.2 Basic Properties of Inequalities

If a, b, c be real numbers. Then

1. $a \geq b$ and $b \geq c \implies a \geq c$
2. $a \geq b$ and $b > c \implies a > c$
3. $a \geq b \implies a + c \geq b + c$
4. $a \geq b$ and $c > 0 \implies ac \geq bc$
5. $a \geq b$ and $c < 0 \implies ac \leq bc$.

Theorem C.1: *If $a_1, a_2, \ldots, a_n$; $b_1, b_2, \ldots, b_n$ be all real numbers such that $a_i > b_i$ for $i = 1, 2, \ldots, n$, then*

$$a_1 + a_2 + \cdots + a_n > b_1 + b_2 + \cdots + b_n.$$

Theorem C.2: *If $a_1, a_2, \ldots, a_n$; $b_1, b_2, \ldots, b_n$ be all positive real numbers such that $a_i > b_i$ for $i = 1, 2, \ldots, n$, then*

$$a_1 a_2 \ldots a_n > b_1 b_2 \ldots b_n.$$

Note: The theorem does not hold if the numbers are not all positive. For example, $3 > -2$ and $4 > -9$ but $3.4 < (-2).(-9)$.

Theorem C.3: *If a, b be positive real numbers with $a > b$, and n be a positive integer, then $a^n > b^n$.*

Note: If a, b are real numbers and $a > b$, and n is a positive integer, it does not necessarily follow that $a^n > b^n$. For example, $2 > -3$ implies $(2)^2 < (-3)^2$, but $2 > -1$ implies $(2)^2 > (-1)^2$.

Theorem C.4: *If a, b be positive real numbers with $a > b$, and n be a negative integer, then $a^n < b^n$.*

Theorem C.5: *If a, b be positive real numbers with $a > b$, and n be a rational number, then $a^n \gtreqless b^n$ according as $n \gtreqless 0$.*

3.3 The Cauchy-Schwarz Inequality

If $a_1, a_2, \ldots, a_n$; $b_1, b_2, \ldots, b_n$ be all real numbers, then

$$(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2 \le (a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2),$$

or in much compact form as

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \le \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right).$$

The equality occurs when either
(i) $a_i = 0$ for $i = 1, 2, \ldots, n$ or $a_i = 0$ for $i = 1, 2, \ldots, n$ or both $a_i = 0$ and $b_i = 0$ for $i = 1, 2, \ldots, n$, or
(ii) $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}$.

3.4 Arithmetic, Geometric, and Harmonic Means

Let $a_1, a_2, \dots, a_n$ be n positive real numbers.
The *arithmetic mean* (A.M.) of the numbers is defined by

$$\text{A.M.} = \frac{a_1 + a_2 + \dots + a_n}{n}.$$

The *geometric mean* (G.M.) of the numbers is defined by

$$\text{G.M.} = \sqrt[n]{a_1 a_2 \dots a_n}.$$

The *harmonic mean* (H.M.) of the numbers is defined by

$$\text{H.M.} = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}}.$$

Let $p_1, p_2, \dots, p_n$ be n positive rational numbers.
The *weighted arithmetic mean* of $a_1, a_2, \dots, a_n$, with associated weights $p_1, p_2, \dots, p_n$ respectively, is defined by

$$A(a,p) = \frac{p_1 a_1 + p_2 a_2 + \dots + p_n a_n}{p_1 + p_2 + \dots + p_n}.$$

The *weighted geometric mean* of $a_1, a_2, \dots, a_n$, with associated weights $p_1, p_2, \dots, p_n$ respectively, is defined by

$$G(a,p) = ({a_1}^{p_1} {a_2}^{p_2} \dots {a_n}^{p_n})^{\frac{1}{p_1 + p_2 + \dots + p_n}}.$$

The *weighted harmonic mean* of $a_1, a_2, \dots, a_n$, with associated weights $p_1, p_2, \dots, p_n$ respectively, is defined by

$$H(a,p) = \frac{p_1 + p_2 + \dots + p_n}{\frac{p_1}{a_1} + \frac{p_2}{a_2} + \dots + \frac{p_n}{a_n}}.$$

Theorem C.6: *If $a_1, a_2, \dots, a_n$ be n positive real numbers, then*

$$\text{A.M.} \geq \text{G.M.} \geq \text{H.M.}.$$

The equality occurs when $a_1 = a_2 = \dots = a_n$.

Theorem C.7: *If $a_1, a_2, \dots, a_n$ be n positive real numbers and $p_1, p_2, \dots, p_n$ be n positive rational numbers, then*

$$A(a,p) \geq G(a,p) \geq H(a,p).$$

The equality occurs when $a_1 = a_2 = \dots = a_n$.

Theorem C.8: *If a be a positive real number, not equal to 1, and m be a rational number, then*

$$a^m - 1 > \text{ or } < m(a-1)$$

according as m does not or does lie between 0 and 1.

Corollary C.8.1: *Let $x > -1$, but not equal to 0 and m be a rational number. Then*

$$(1+x)^m > \text{ or } < 1 + mx$$

according as m does not or does lie between 0 and 1.

Corollary C.8.2: *Let $x < 1$, but not equal to 0 and m be a rational number. Then*

$$(1-x)^m > \text{ or } < 1 - mx$$

according as m does not or does lie between 0 and 1.

Corollary C.8.3: *Let a and b be unequal positive numbers and m be a rational number. Then*

$$ma^{m-1}(a-b) > \text{ or } < a^m - b^m > \text{ or } < mb^{m-1}(a-b)$$

according as m does not or does lie between 0 and 1.

Theorem C.9: *If $a_1, a_2, \ldots, a_n$ be n positive real numbers, not all equal, and m be a rational number, then*

$$\frac{{a_1}^m + {a_2}^m + \cdots + {a_n}^m}{n} > \text{ or } < \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^m$$

according as m does not or does lie between 0 and 1. The equality occurs when $a_1 = a_2 = \cdots = a_n$.

Theorem C.10: *If $a_1, a_2, \ldots, a_n$ be n positive real numbers, not all equal, $p_1, p_2, \ldots, p_n$ be positive real numbers and m be a rational number, then*

$$\frac{p_1{a_1}^m + p_2{a_2}^m + \cdots + p_n{a_n}^m}{p_1 + p_2 + \cdots + p_n} > \text{ or } < \left(\frac{p_1a_1 + p_2a_2 + \cdots + p_na_n}{p_1 + p_2 + \cdots + p_n}\right)^m$$

according as m does not or does lie between 0 and 1. The equality occurs when $a_1 = a_2 = \cdots = a_n$.

Corollary C.10.1: *If $a_1, a_2, \ldots, a_n$ be n positive real numbers, not all equal, $q_1, q_2, \ldots, q_n$ be positive real numbers and m be a rational number, then*

$$q_1{a_1}^m + q_2{a_2}^m + \cdots + q_n{a_n}^m > \text{ or } < (q_1a_1 + q_2a_2 + \cdots + q_na_n)^m$$

according as m does not or does lie between 0 *and* 1*. The equality occurs when $a_1 = a_2 = \cdots = a_n$.*

3.5 Applications to Problems of Maxima and Minima

1. Let $x_1, x_2, \ldots, x_n$ be n positive real numbers such that their sum $x_1 + x_2 + \cdots + x_n = k$, a constant. Then by **Theorem C.6**,

$$\left(\frac{k}{n}\right)^n \geq x_1 x_2 \ldots x_n.$$

Therefore, the *maximum* value of $x_1 x_2 \ldots x_n$ occurs when $x_1 = x_2 = \cdots = x_n = \frac{k}{n}$ and the maximum value is $\left(\frac{k}{n}\right)^n$.

2. Let $x_1, x_2, \ldots, x_n$ be n positive real numbers such that their product $x_1 x_2 \ldots x_n = k$, a constant. Then by **Theorem C.6**,

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq k^{\frac{1}{n}}.$$

Therefore, the *minimum* value of $x_1 x_2 \ldots x_n$ occurs when $x_1 = x_2 = \cdots = x_n = \frac{k}{n}$ and the maximum value is $nk^{\frac{1}{n}}$.

3. Let $x_1, x_2, \ldots, x_n$ be n positive real numbers such that their sum $x_1 + x_2 + \cdots + x_n = k$, a constant, and m be a rational number other than 0 and 1. Then by **Theorem C.9**,

$$\begin{aligned} \frac{{x_1}^m + {x_2}^m + \cdots + {x_n}^m}{n} &\geq \left(\frac{k}{n}\right)^m \text{ when } m > 1 \text{ or } m < 0; \\ \frac{{x_1}^m + {x_2}^m + \cdots + {x_n}^m}{n} &\leq \left(\frac{k}{n}\right)^m \text{ when } 0 < m < 1. \end{aligned}$$

Therefore, when $m > 1$ or $m < 0$, the *minimum* value of ${x_1}^m + {x_2}^m + \cdots + {x_n}^m$ occurs when $x_1 = x_2 = \cdots = x_n = \frac{k}{n}$ and the minimum value is $n\left(\frac{k}{n}\right)^m$. On the other hand, When $0 < m < 1$, the *maximum* value of ${x_1}^m + {x_2}^m + \cdots + {x_n}^m$ occurs when $x_1 = x_2 = \cdots = x_n = \frac{k}{n}$ and the maximum value is $n\left(\frac{k}{n}\right)^m$.

4. Let $x_1, x_2, \ldots, x_n$ be n positive real numbers such that ${x_1}^m + {x_2}^m + \cdots + {x_n}^m = k$, a constant, and m be a rational number other than 0 and 1. Then by **Theorem C.9**,

$$\begin{aligned} \frac{k}{n} &\geq \left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)^m \text{ when } m > 1 \text{ or } m < 0; \\ \frac{k}{n} &\leq \left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)^m \text{ when } 0 < m < 1. \end{aligned}$$

Hence,

$$\begin{aligned}
\left(\frac{k}{n}\right)^{\frac{1}{m}} &\geq \left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)^m \text{ when } m > 1; \\
\left(\frac{k}{n}\right)^{\frac{1}{m}} &\leq \left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)^m \text{ when } 0 < m < 1; \\
\left(\frac{k}{n}\right)^{\frac{1}{m}} &\leq \left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)^m \text{ when } m < 0.
\end{aligned}$$

Therefore, when $m > 1$, the *maximum* value of $x_1+x_2+\cdots+x_n$ occurs when $x_1 = x_2 = \cdots = x_n = \left(\frac{k}{n}\right)^{\frac{1}{m}}$ and the maximum value is $n\left(\frac{k}{n}\right)^{\frac{1}{m}}$. On the other hand, when $m < 0$ or $0 < m < 1$, the *minimum* value of $x_1 + x_2 + \cdots + x_n$ occurs when $x_1 = x_2 = \cdots = x_n = \left(\frac{k}{n}\right)^{\frac{1}{m}}$ and the minimum value is $n\left(\frac{k}{n}\right)^{\frac{1}{m}}$.

Note that the above applications can be extended to include those cases also where weights are taken into account.